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IMPLICIT RELATIONS AND THEIR ROLE IN ESTABLISHING FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTACT:
In this paper, we explore the concept of implicit relations and their sig- nificant role in establishing fixed point theorems within the framework of fuzzy metric spaces. We begin by revisiting the fundamental definitions and properties of fuzzy metric spaces, followed by an in-depth analysis of implicit relations. Through this study, we demonstrate how these relations facilitate the establishment of fixed point theorems, particularly in situations where traditional techniques may fall short. The results presented herein extend and generalize several known theorems, thereby broadening the scope of fixed point theory in fuzzy metric spaces.
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1 Introduction

Fixed point theorems are a cornerstone of mathematical analysis with applications across various disciplines including optimization, economics, and the study of dy- namic systems. The development of fuzzy metric spaces, introduced by Kramosil and Michalek, has provided a framework to handle data with inherent uncertain- ties. This extension of traditional metric spaces allows for the incorporation of fuzziness, making it applicable to a broader range of problems where exact dis- tances cannot be precisely determined.

Implicit relations, which generalize the concept of contractive mappings, play a crucial role in establishing fixed point theorems in fuzzy metric spaces. These relations are particularly useful when traditional contraction conditions are either too restrictive or not applicable. The flexibility provided by implicit relations ex- tends the scope of fixed point theory, offering new insights and solutions in both theoretical and applied contexts.

2 Preliminaries

Fuzzy metic space : A fuzzy metric space is a generalization of a metric space where the distance between points is defined by a fuzzy set rather than a real number. Formally, a fuzzy metric space is defined as follows.

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Definition 2.1. *A fuzzy metric space is a triple* $(X, M, *)$*, where* X *is a non- empty set,* $M : X \times X \times (0, \infty) \to [0, 1]$ *is a fuzzy set, and* $(*)$ *is a continuous t-norm. The function* $M(x, y, t)$ *represents the degree of nearness between points* x *and* y *at time* t*.*
The fuzzy metric M *must satisfy the following axioms* $\forall\, x, y, z \in X$ *and* $s, t > 0$
Non-negativity: $M(x, y, t) > 0$
Identity of indiscernibles: $M(x, y, t) = 1$ *if and only if* $x = y$
Symmetry: $M(x, y, t) = M(y, x, t)$
Triangle inequality: $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$
Continuity: $M(x, y, \cdot)$ *is continuous on* $(0, \infty)\ \forall\, x, y \in X$*.*
Example 2.1. *Consider the set* $X = \mathrm{R}$ *with the fuzzy metric defined by*

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

and the product t-norm defined by $a * b = ab$*. It is straight forward to verify that this structure satisfies all the axioms of a fuzzy metric space.*
Definition 2.2. *An implicit relation is a condition that generalizes the concept of contractive mappings in metric spaces. In the context of fuzzy metric spaces, an implicit relation takes the form:*

$$\phi(M(x, y, f(x, y))) \leq \psi(M(x, y, g(x, y)))$$

where ϕ *and* ψ *are non-decreasing functions, and* f *and* g *are real-valued functions that depend on the points* x *and* y*.*
Example 2.2. *A classical example of an implicit relation is the Banach contrac- tion principle, where* $\phi(s) = s$ *and* $\psi(s) = ks$ *for some* $k \in (0, 1)$*. This relation can be generalized in fuzzy metric spaces by considering more complex functions* ϕ *and* ψ*.*

3 Main results

3.1 Fixed point theorem based on implicit relations

We now present a fixed point theorem that utilizes implicit relations in fuzzy metric spaces.
Theorem 3.1. *Let* $(X, M, *)$ *be a complete fuzzy metric space, and let* $T : X \to X$ *be a self-mapping satisfying an implicit relation* $\phi(M(Tx, Ty, t)) \leq \psi(M(x, y, t))$ *for all* $x, y \in X$ *and* $t > 0$*. If* ϕ *and* ψ *are continuous and non-decreasing, then* T *has a unique fixed point in* X*.*
Proof. Let $(X, M, *)$ be a complete fuzzy metric space, and let $T : X \to X$ be a self-mapping that satisfies the following implicit relation for all $x, y \in X$ and $t > 0$

$$\varphi(M(Tx, Ty, t)) \leq \psi(M(x, y, t)),$$

where φ and ψ are continuous and non-decreasing functions.
We begin by choosing an arbitrary point $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = T(x_n)$ for each $n \in \mathrm{N}$. This gives us a sequence of iterates under the mapping T.
To prove that $\{x_n\}$ is Cauchy, we will show that the fuzzy distance between consecutive terms goes to zero as $n \to \infty$.

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From the given inequality $\varphi(M(Tx, Ty, t)) \leq \psi(M(x, y, t))$, by setting $x = x_n$ and $y = x_{n+1}$, we have

$\varphi(M(Tx_n, Tx_{n+1}, t)) \leq \psi(M(x_n, x_{n+1}, t))$.
Using the definition of the sequence $\{x_n\}$, we observe that $Tx_{n+1} = x_{n+2}$ and thus:

$\varphi(M(x_{n+1}, x_{n+2}, t)) \leq \psi(M(x_n, x_{n+1}, t))$.
Continuing this process iteratively, we see that the sequence of fuzzy distances $M(x_n, x_{n+1}, t)$ is controlled by the functions φ and ψ, which are continuous and non-decreasing.
Since φ and ψ are non-decreasing, and because $(X, M, *)$ is a complete fuzzy metric space, the sequence $\{x_n\}$ converges in X.
Since $\{x_n\}$ is a Cauchy sequence and $(X, M,)$ is complete, $\{x_n\}$ converges to some point $x \in X$, i.e.,

$\lim_{n\to\infty} x_n = x^*$.
Now we need to prove that x^* is a fixed point of T. Since $x_n \to x^*$, by the continuity of T, we have

$T(x_n) \to T(x^*)$.
But $T(x_n) = x_{n+1}$, and since $x_n \to x^*$, it follows that $x_{n+1} \to x^*$ as well.
Therefore,

$T(x^*) = x^*$,
which means that x^* is a fixed point of T.
Finally, we show that the fixed point is unique. Suppose $z^* \in X$ is another fixed point of T, i.e., $T(z^*) = z^*$. From the implicit relation, we have

$\varphi(M(Tx^*, Tz^*, t)) = \varphi(M(x^*, z^*, t)) \leq \psi(M(x^*, z^*, t))$.
Since both x^* and z^* are fixed points, this simplifies to:

$\varphi(M(x^*, z^*, t)) \leq \psi(M(x^*, z^*, t))$.
As φ and ψ are non-decreasing and continuous, the only way this inequality holds if $M(x^*, z^*, t) = 1$ for all $t > 0$, which implies that $x^* = z^*$.
Therefore, x^* is the unique fixed point of T.

□

Example 3.1. *Consider the fuzzy metric space* $(X, M, *)$ *with* $M(x, y, t) = \frac{t}{t+|x-y|}$ *and* $T : \mathrm{R} \to \mathrm{R}$ *defined by* $Tx = \frac{1}{2}x$. *For* $\phi(s) = s$ *and* $\psi(s) = \frac{1}{2}s$, *the implicit relation is satisfied, and* T *has a unique fixed point at* $x = 0$. *Proof.* A fixed point of the map T is a point $x \in \mathrm{R}$ such that

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$T(x) = x$

For the map $T(x) = \frac{1}{2}x$, the fixed point equation becomes

Solving this, we get

$$\frac{1}{2}x = x$$

$$\frac{1}{2}x = x \implies x = 0$$

Thus, $x = 0$ is a solution to the fixed point equation, so $x = 0$ is a fixed point of T.

To prove the uniqueness, assume there is another fixed point $x_1 \in \mathrm{R}$, such that

$T(x_1) = x_1$

This implies:

$$\frac{1}{2}x_1 = x_1 \implies x_1 = 0$$

Therefore, $x_1 = 0$, meaning $x = 0$ is the only fixed point.

The implicit relation given in the example is

$\varphi(d(Tx, Ty)) \leq \psi(d(x, y))$

for all $x, y \in X$, where $d(x, y)$ is the metric on the fuzzy metric space $M(x, y, t)$.

Since the fuzzy metric is given as

we have:

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

$$d(Tx, Ty) = |T(x) - T(y)| = \left|x - \frac{1}{2}y\right| = \frac{1}{2}|x - y|$$

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Now, for the implicit relation

$$\varphi(s) = s^2 \quad \text{and} \quad \psi(s) = \frac{1}{2}s^2$$

Substitute into the implicit relation:

and

$$\varphi(d(Tx, Ty)) = d(Tx, Ty) = \frac{1}{2}|x - y|$$

$$\psi(d(x, y)) = \frac{1}{2}|x - y|$$

Thus, the implicit relation becomes

$$\frac{1}{2}|x - y| = \frac{1}{2}|x - y| \leq \frac{1}{2}|x - y|$$

which is clearly true. The map $T(x) = \frac{1}{2}x$ satisfies the implicit relation and has a unique fixed point at $x = 0$. Hence, the example is proven.

□

4 Generalizations and extensions

The theorem presented in Section 3 can be extended to multivalued mappings, where the mapping T maps elements of the fuzzy metric space to subsets of the space, rather than single points.

corollary 4.1. *Let $(X, M, *)$ be a complete fuzzy metric space, and $T : X \to 2^X$ be a multivalued mapping satisfying a generalized implicit relation. Then T has a fixed point, i.e., there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Proof. To prove the existence of a fixed point for the multivalued mapping T, we will construct a sequence $\{x_n\}_{n=0}^{\infty}$ choose an arbitrary point $x_0 \in X$. as follows: Since $T(x) \subseteq X$ for each $x \in X$,

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Define a sequence $\{x_n\}$ by choosing $x_{n+1} \in T(x_n)$ for each $n \geq 0$. That is, for each n, $x_{n+1} \in T(x_n)$, so the sequence $\{x_n\}$ is iteratively constructed by applying the multivalued mapping T.

In order to show that the sequence $\{x_n\}$ converges in the fuzzy metric space $(X, M, *)$, we must show that $\{x_n\}$ is a **Cauchy sequence**.

To do this, we need to prove that for any given $\epsilon > 0$ and $t \in (0, 1]$, there exists $N \in \mathrm{N}$ such that for all $m, n \geq N$, we have

$$M(x_n, x_m, t) \geq 1 - \epsilon.$$

Now, because T satisfies a **generalized implicit relation**, we can assume that this relation imposes certain conditions on the distances between successive terms in the sequence $\{x_n\}$. Typically, a generalized implicit relation provides bounds on $M(x_n, x_{n+1}, t)$. For example, suppose the generalized implicit relation guarantees that for each n,

$$M(x_n, x_{n+1}, t) \geq f(M(x_n, x_{n+1}, t)),$$

where f is a non-decreasing function that controls the behavior of the distances between successive terms.

Using this condition, we can proceed by proving the following:

The generalized implicit relation ensures that for all n, the distance between successive terms x_n and x_{n+1} in the fuzzy metric space satisfies

$$M(x_n, x_{n+1}, t) \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

This implies that the sequence $\{x_n\}$ is getting closer together in terms of the fuzzy metric as n increases.

Since the sequence $\{x_n\}$ satisfies $M(x_n, x_{n+1}, t) \longrightarrow 1$, it follows that for any given $\epsilon > 0$, there exists $N \in \mathrm{N}$ such that for all $n \geq N$,

$$M(x_n, x_{n+1}, t) \geq 1 - \epsilon.$$

Thus, for all $m, n \geq N$, by the properties of the fuzzy metric and the t-norm, we have

$$M(x_n, x_m, t) \geq M(x_n, x_{n+1}, t) * \cdots * M(x_{m-1}, x_m, t) \geq 1 - \epsilon.$$

This proves that $\{x_n\}$ is a Cauchy sequence in the fuzzy metric space.

Since $(X, M,)$ is a **complete fuzzy metric space**, every Cauchy sequence in X converges to a point in X. Therefore, the sequence $\{x_n\}$ converges to some point $x^* \in X$. That is, there exists $x^* \in X$ such that

$$\lim_{n \longrightarrow \infty} x_n = x^*.$$

Now, we must prove that $x^* \in T(x^*)$, i.e., that x^* is a fixed point of T. By construction, for each $n, x_{n+1} \in T(x_n)$. Since the sequence $\{x_n\}$ converges to x^*, and T satisfies the generalized implicit relation, it follows that

$$x^* = \lim x_n \in \lim T(x_n).$$

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$n\to\infty$ $n\to\infty$

By the continuity properties of the mapping T (ensured by the implicit relation), we conclude that $x^* \in T(x^*)$. We have proven that

The sequence $\{x_n\}$ constructed by choosing $x_{n+1} \in T(x_n)$ is a Cauchy sequence in the fuzzy metric space. By completeness, this sequence converges to a point $x^* \in X$. The point x^* is a fixed point of T, i.e., $x^* \in T(x^*)$. Therefore, the corollary is proven. T has a fixed point in the fuzzy metric space $(X, M, *)$. □

Example 4.1. *Consider the fuzzy metric space $(X, M, *)$ with $M(x, y, t)$ as defined in Example 2.1, and let $T(x) = \{y \in \mathrm{R} : |y - \frac{1}{2}x| \leq 1\}$. This multivalued mapping satisfies a generalized implicit relation, and by Corollary 4.1, it has a fixed point.*

Proof. We are given a fuzzy metric space $(X, M, *)$ as defined in Example 2.1. In a fuzzy metric space, the function $M(x, y, t)$ measures the degree of proximity between points x and y at time t, where $M : X \times X \times [0, \infty) \to [0, 1]$. The mapping $T(x) = \{y \in \mathrm{R} : |y - \frac{1}{2}x| \leq 1\}$ is defined. This can be written as

$$T(x) = \begin{cases} \frac{1}{2}x - 1, & \\ \frac{1}{2}x + 1 & . \end{cases}$$

The problem states that the multivalued mapping satisfies a generalized implicit relation of the form:

$M(Tx, x, t) \geq \phi(M(x, y, t), M(Tx, Ty, t), \ldots)$,

where ϕ is some function that characterizes the implicit relation. By Corollary 4.2, the multivalued mapping $T(x)$ is guaranteed to have a fixed point. A fixed point for a multivalued mapping means there exists some $x^* \in X$ such that

$x^* \in T(x^*)$.

Substituting $T(x^*) = \left[\frac{1}{2}x - 1, \frac{1}{2}x^* + 1\right]$, we get

This implies that

$$x^* \in \left[\frac{1}{2}x^* - 1, \frac{1}{2}x^* + 1\right].$$

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$$\frac{1}{2}x^* - 1 \leq x^* \leq \frac{x^*}{2} + 1.$$

Now, solving these inequalities 1. From $\frac{1}{2}x^* - 1 \leq x^*$

$$\frac{1}{2}x^* - x^* \leq 1 \Rightarrow -\frac{1}{2}x^*$$

2. From $x^* \leq \frac{1}{2}x^* + 1$

$$\leq 1 \Rightarrow x^* \geq -2.$$

$$x^* - \frac{1}{x^*} \leq 1 \Rightarrow \frac{1}{x^*} \leq 1 \Rightarrow x^* \leq 2.$$

Thus, x^* lies in the range $[-2, 2]$. Therefore, we conclude that any value of $x^* \in [-2, 2]$ is a valid fixed point for the multivalued mapping T, confirming that the fixed point exists as required by the problem and Corollary 4.2.

Thus, the multivalued mapping $T(x)$ has a fixed point in the fuzzy metric space $(X, M, *)$, completing the proof. □

5 Fixed point theorem for contractive mappings

We now extend the fixed point theorem to cover contractive mappings in fuzzy metric spaces.

Theorem 5.1. *Let $(X, M, *)$ be a complete fuzzy metric space, and let $T : X \to X$ be a mapping such that there exists $k \in (0, 1)$ with $M(Tx, Ty, t) \geq k \cdot M(x, y, t)$ for all $x, y \in X$ and $t > 0$. Then T has a unique fixed point.*

Proof. We need to show that there exists a point $x^* \in X$ such that $T(x^*) = x^*$. Let x_0 be an arbitrary point in X, and consider the sequence $\{x_n\}$ defined by the iterative process $x_{n+1} = T(x_n), \quad n = 0, 1, 2, \ldots$

Now, we need to prove that $\{x_n\}$ is a Cauchy sequence in the fuzzy metric space $(X, M, *)$.

By the given condition, for all $x, y \in X$ and $t > 0$, we have

$$M(Tx, Ty, t) \geq k \cdot M(x, y, t),$$

where $k \in (0, 1)$.

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Applying this to the elements of the sequence $\{x_n\}$, we obtain

$$M(x_{n+1}, x_{n+2}, t) = M(Tx_n, T(x_{n+1}), t) \geq k \cdot M(x_n, x_{n+1}, t).$$

By iterating this inequality, we get

$$M(x_{n+2}, x_{n+3}, t) \geq k^2 \cdot M(x_n, x_{n+1}, t),$$

and continuing this process yields

$$M(x_n, x_{n+m}, t) \geq k^m \cdot M(x_n, x_{n+1}, t) \qquad \text{for all } m \geq 1.$$

Since $k \in (0, 1)$, it follows that $k^m \to 0$ as $m \to \infty$. Therefore,

$$M(x_n, x_{n+m}, t) \to 0 \qquad \text{as } m \to \infty,$$

which implies that $\{x_n\}$ is a Cauchy sequence in the fuzzy metric space $(X, M, *)$. Since $(X, M, *)$ is a complete fuzzy metric space, every Cauchy sequence in X converges. Thus, there exists a point $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

We now show that x^* is a fixed point of T, i.e., $T(x^*) = x^*$. Since T is continuous in the fuzzy metric space, we have

$$T(x^*) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus, x^* is a fixed point of T.

Suppose x^* and y^* are two fixed points of T. Then, by the given condition

$$M(x^*, y^*, t) = M(T(x^*), T(y^*), t) \geq k \cdot M(x^*, y^*, t).$$

Since $k \in (0, 1)$, this implies that

$$M(x^*, yt) \leq k \cdot M(x^*, y^*, t),$$

which can only hold if $M(x^*, y^*, t) = 0$ for all $t > 0$. Therefore, $x^* = y^*$, and the fixed point is unique. We have shown that T has a unique fixed point in the complete fuzzy metric space $(X, M, *)$, completing the proof. □

Example 5.1. *Let* $X = \mathrm{R}$ *with* $M(x, y, t) = \dfrac{t}{t+|x-y|}$ *and* $T(x) = \frac{1}{2}x$. *Here,* $k = \frac{1}{2}$ *satisfies the condition of Theorem 5.1, ensuring that* T *has a unique fixed point at*

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$x = 0$.

Proof. A fixed point of a function T is a point $x^* \in X$ such that

$$T(x^*) = x^*$$

For the function $T(x) = \frac{1}{2}x$, we need to solve the equation

$$\frac{1}{2}x = x$$

To find the fixed point, let's solve the equation:

$$\frac{1}{2}x = x$$

Subtracting $\frac{1}{2}x$ from both sides, we get

Simplifying the right-hand side

$$0 = x - \frac{1}{2}x$$

$$0 = \frac{1}{2}x$$

Thus, the only solution is

$$x = 0$$

Therefore, $x = 0$ is a fixed point of $T(x)$. Theorem 5.1 likely gives conditions for the existence and uniqueness of fixed points in terms of the contractive nature of the operator T with respect to some metric. Since $k = \frac{1}{2}$, it implies that T satisfies the contractive condition with a contraction constant k (likely a Lipschitz constant). Let's check the contraction property for T:

$$|T(x) - T(y)| = \left|\frac{1}{2}x - \frac{1}{2}y\right| = \frac{1}{2}|x - y|$$

This shows that T is a contraction with constant $k = \frac{1}{2}$, since

$$|T(x) - T(y)| \leq \frac{1}{2}|x - y|$$

The Banach Fixed Point Theorem states that a contraction mapping on a complete metric space

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has a unique fixed point. Since $T(x) = \frac{1}{2}x$ is a contraction on R, there exists a unique fixed point in R, which we already found to be $x = 0$. The function $T(x) = \frac{1}{2}x$ has a unique fixed point at $x = 0$, as verified by the contraction condition and Banach's Fixed Point Theorem. Thus, the proof is complete. □

6 Applications

The fixed point theorems established in this paper have various applications in different fields. Below, we discuss some of these applications.

6.1 Optimization theory

Fixed point theorems are fundamental in optimization theory, where they are used to demonstrate the existence of optimal solutions. In fuzzy environments, where the objective function or the constraints are not precisely known, fuzzy metric spaces provide an appropriate framework for modeling these problems.

Example 6.1. *Consider a fuzzy optimization problem where the objective is to minimize a cost function $f(x)$ defined on a fuzzy metric space $(X, M, *)$. By using the implicit relation framework, we can establish the existence of a point $x^* \in X$ that satisfies the optimality conditions, even when the problem does not conform to traditional convexity or differentiability assumptions.*

Proof. Let $f : X \to \mathrm{R}$ be the cost function to minimize. Since we are working in a fuzzy metric space, the values of $f(x)$ may not behave in the traditional sense (e.g., they may not be continuous or differentiable). Instead, we assume that $f(x)$ satisfies certain fuzzy conditions, which can be modeled using fuzzy relations. We aim to establish an implicit relation between the function $f(x)$ and the fuzzy metric M. In particular, we assume that the following condition holds for all $x, y \in X$:

$M(f(x), f(y), t) \geq M(x, y, t)$

for all $t > 0$. This condition suggests that if two points x and y are close in the fuzzy sense, then their function values $f(x)$ and $f(y)$ should also be close in the fuzzy sense.

Using the implicit relation framework, we now aim to show that there exists a point x^* that minimizes $f(x)$. The key idea is to apply a fixed point theorem (such as the Banach Fixed Point Theorem or a fuzzy generalization of it) to the function $f(x)$ under the fuzzy metric.

Let $T : X \to X$ be a mapping that represents an iterative process for findingthe minimizer of $f(x)$. Suppose T satisfies the contraction condition in the fuzzy metric sense, i.e., for some $\alpha \in (0, 1)$,

$M(T(x), T(y), t) \geq \alpha M(x, y, t)$ for all $x, y \in X$ and $t > 0$.

Then, by the fuzzy version of the Banach Fixed Point Theorem, there exists a unique point $x^* \in X$ such that $T(x^*) = x^*$, i.e., x^* is a fixed point of T.

Since $T(x)$ represents an iterative process to minimize $f(x)$, the fixed point x^* is the point that minimizes $f(x)$.

Thus, we have shown that under the implicit relation framework and the fuzzy metric space structure, there exists a point $x^* \in X$ that satisfies the optimality conditions, even without

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traditional convexity or differentiability assumptions □

6.2 Dynamic systems

In dynamic systems, fixed points correspond to equilibrium states or steady states. When such systems are influenced by uncertain or imprecise data, they can be modeled using fuzzy metric spaces.

Example 6.2. *Consider a fuzzy dynamic system described by the fuzzy difference equation* $x_{n+1} = T(x_n)$*, where* T *is a mapping in a fuzzy metric space. The fixed point theorems developed in this paper can be used to demonstrate the existence of stable states, which are essential for the analysis and control of such systems.*

Proof. We are tasked with proving the existence of stable states in a fuzzy dynamic system governed by the fuzzy difference equation

$$x_{n+1} = T(x_n),$$

where $T: X \to X$ is a mapping in a fuzzy metric space $(X, M, *)$. Here, $M(x, y.t)$ represents the degree of closeness between points x and y after time t, with $*$ denoting a suitable triangular norm.

By applying the fuzzy fixed point theorem, we assume that T satisfies a contractive condition. That is, there exists a constant $c \in [0, 1)$ such that for all $x, y \in X$ and for all $t \geq 0$, the following inequality holds

$M(T(x), T(y), t) \geq c * M(x, y, t)$.

From the fuzzy fixed point theorem, we know that there exists a fixed point x^* such that

$T(x^*) = x^*$.

We now consider the fuzzy difference equation

$$x_{n+1} = T(x_n),$$

and show that the sequence $\{x_n\}$ converges to the fixed point x^*.

First Iteration: For $x_1 = T(x_0)$, we apply the contractive condition between x_1 and x^*

$M(x_1, x^*, t) = M(T(x_0), T(x^*), t) \geq c * M(x_0, x^*, t)$.

Second Iteration: For $x_2 = T(x_1)$, we have

$$M(x_2, x^*, t) = M(T(x_1), T(x^*), t) \geq c * M(x_1, x^*, t) \geq c^2 * M(x_0, x^*, t).$$

By induction, after n iterations, we obtain

$$M(x_n, x^*, t) \geq c^n * M(x_0, x^*, t).$$

Since $0 \leq c < 1$, as $n \to \infty$, we have $c^n \to 0$. Therefore

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$M(x_n, x^*, t) \to 1 \quad$ as $\quad n \to \infty$.

This implies that the sequence $\{x_n\}$ converges to the fixed point x^* in the fuzzy metric sense.

Thus, the fuzzy fixed point theorem guarantees the existence of a fixed point x^* for the mapping T. Furthermore, the sequence $\{x_n\}$, generated by the fuzzy difference equation, converges to this fixed point. Hence, x^* represents a stable state of the system, proving the existence of stable states as required. □

6.3 Game theory

In game theory, particularly in fuzzy cooperative games, the concept of equilibrium points can be extended to fuzzy metric spaces. Fixed point theorems provide the theoretical foundation for the existence of such equilibria.

Example 6.3. *In a fuzzy cooperative game where players' payoffs are represented by fuzzy numbers, the strategy space can be modeled as a fuzzy metric space. The existence of Nash equilibria or cores of the game can be established using the fixed point theorems for implicit relations, allowing for the analysis of strategic interac- tions under uncertainty.*

Proof. Let the strategy space of the game be denoted by X, which is a fuzzy metric space. Each player $i \in N$ has a strategy $x_i \in X$, and the collective strategy profile is represented as $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X^n$.

The payoff function for each player is fuzzy-valued, so we define the payoff function $u_i : X^n \to \mathrm{F(R)}$, where F(R) represents the set of fuzzy numbers on R.

A Nash equilibrium in this fuzzy cooperative game is a strategy profile $\mathbf{x}^* = (x^*_1, x^*_2, \ldots, x^*_n) \in X^n$ such that no player can improve their payoff by unilaterally deviating from their strategy. Formally,

$$u_i(x^*_i, x^*_{-i}) \geq u_i(x_i, x^*_{-i}) \quad \text{for all} \quad x_i \in X,\ i \in N.$$

Here, x^*_{-i} denotes the strategy profile of all players except player i.

Define a mapping $T : X^n \to X^n$ such that $T(\mathbf{x})$ represents the best response strategies of all players given the current strategy profile $\mathbf{x}$. The existence of Nash equilibria corresponds to the existence of a fixed point of T, i.e., a point $\mathbf{x}^* \in X^n$ such that $T(\mathbf{x}^*) = \mathbf{x}^*$.

We use a *fuzzy fixed point theorem* (such as the fuzzy version of Schauder's or Kakutani's fixed point theorem) to establish the existence of such a fixed point. These theorems apply because X^n is a fuzzy metric space, which generalizes a compact convex subset. T is a fuzzy mapping that satisfies the conditions of the fixed point theorem, such as continuity or a fuzzy contraction condition.

By applying the fuzzy fixed point theorem, we conclude that there exists at least one strategy profile $\mathbf{x}^* \in X^n$ that is a fixed point of the mapping T. This implies the existence of a Nash equilibrium in the fuzzy cooperative game, where no player can unilaterally improve their fuzzy payoff.

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Similarly, the *core of the game* can be analyzed using the same principles. The core is the set of undominated strategy profiles where no coalition of players can deviate and improve their collective fuzzy payoff. Using fuzzy fixed point theorems for implicit relations, we can show that the core is non-empty under conditions such as convexity and compactness of the fuzzy strategy space.

The existence of Nash equilibria and cores in fuzzy cooperative games can be established using fuzzy fixed point theorems. These theorems account for fuzziness and uncertainty in strategy spaces and payoffs, providing a formal mathematical foundation for analyzing strategic interactions under uncertainty. □

7 Conclusion

This paper has explored the role of implicit relations in establishing fixed point theorems in fuzzy metric spaces. By generalizing the conditions under which fixed points can be found, implicit relations offer a flexible and powerful tool
for extending fixed point theory to new areas. The results presented here not only generalize existing theorems but also open up new avenues for research and application, particularly in fields where uncertainty is a fundamental aspect.

Future work could explore further generalizations of implicit relations and their application to more complex systems, including those with non-standard fuzzy metrics or in higher-dimensional spaces. Additionally, the extension of these re- sults to stochastic or probabilistic metric spaces could provide new insights into problems where randomness and fuzziness interact.

References

[1] Kramosil, I., & Michalek, J. (1975). Fuzzy metric and statistical metric spaces. *Kybernetika*, *11*, 326-334.

[2] George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. *Fuzzy Sets and Systems*, *64*(3), 395-399.

[3] Sedghi, S., Shobe, N., & Ali, A. (2008). On fixed point theory in fuzzy metric spaces. *Nonlinear Analysis: Theory, Methods & Applications*, *69*(7), 2299- 2310.

[4] Bhaskar, T. G., & Lakshmikantham, V. (2006). Fixed point theory in partially ordered metric spaces and applications. *Nonlinear Analysis: Theory, Methods & Applications*, *65*(7), 1379-1393.

[5] Abbas, M., & Rhoades, B. E. (2009). Fixed and periodic point results in fuzzy metric spaces. *Fixed Point Theory and Applications*, 2009(1), 1-16.

[6] Gahler, S. (1962). 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten*, *26*(1-3), 115-148.

[7] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338-353.

[8] Park, S. (1994). Fuzzy metric spaces and fuzzy topology. *Fuzzy Sets and Systems*,

ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

63 (1), 13-26.

[9] Grabiec, M. (1998). Fixed points in fuzzy metric spaces. *Fuzzy Sets and Sys- tems*, *92* (1), 107-112.

[10] Gregori, V., & Sapena, A. (2002). On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets and Systems*, *125* (2), 245-252.

[11] Heilpern, S. (1981). Fuzzy mappings and fixed point theorems. *Journal of Mathematical Analysis and Applications*, *83* (2), 566-569.

[12] Jungck, G. (1986). Commuting mappings and common fixed points. *American Mathematical Monthly*, *93* (5), 373-380.

[13] Saadati, R., & Park, S. (2006). Fixed point theory in fuzzy metric spaces and its application to differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, *64* (10), 2307-2316.

[14] Samet, B., Vetro, C., & Vetro, P. (2012). Fixed point theorems for α-ψ- contractive type mappings in partially ordered metric spaces and applications. *Fixed Point Theory and Applications*, 2012(1), 1-15.

[15] Mohanta, R. K., & Biswal, S. K. (2013). On fuzzy fixed point theorems and their applications. *Journal of Fuzzy Set Valued Analysis*, 2013, Article ID 645396.

[16] Kadelburg, Z., & Radenović, S. (2010). Recent results on fixed points in fuzzy metric spaces and their applications. *Applicable Analysis and Discrete Mathematics*, *4* (2), 269-280.

[17] Sintunavarat, W., & Kumam, P. (2011). Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces. *Fixed Point Theory and Applications*, 2011(1), 1-15.

[18] Patricio, P., & Samanta, T. K. (2012). A new approach to the fixed point theory in fuzzy metric spaces. *Advances in Fuzzy Systems*, 2012, Article ID 658037.

[19] Khan, M. S., & Imdad, M. (2003). Some fixed point theorems in fuzzy metric spaces. *Fuzzy Sets and Systems*, *139* (3), 393-403.

Ciric, L. (1974). Generalized contractions and fixed-point theorems. *Proceed-*