ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

SEPARATION AXIOMS THROUGH βws – CLOSED SETS IN TOPOLOGICAL SPACES

***1 S. Saranya,*2 V.E. Sasikala**

*[1] Research Scholar, *[2] Corresponding Author, Research Supervisor, Associate Professor, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies, Pallavaram, Chennai. India.

Corresponding Author: sasikala.sbs@vistas.ac.in

Abstract:

In this research paper presents an innovative class of Beta weakly semi-closed sets namely, Separation axioms through βws – closed sets in topological spaces. Throughout this paper, $\beta ws - T_0$ space (i = 0,1,2..) were examined to get the fundamental facts in the Beta weakly semi-closed sets. We study the relationship among themselves and with known Separation axioms. Here, many characterizations were obtained along with some of their feature results.

KEY WORDS:

$\beta ws - T_0$ [Beta weakly semi-closed sets], $\beta ws - T_1$, $\beta ws - T_2$.

1. INTRODUCTION:

The fundamental concepts of separation axioms play crucial roles in general topology and various other branches of mathematics. Numerous researchers have investigated exploring their fundamental properties, leading to inspirations for generalizing these concepts to innovative extents. [1] Alias B. Khalaf and Suzan N. Dawod established the concept of g*b-separation axioms in 2013. Many authors from all across the world have contributed to this research. [2] Devi, R. Balachandran, K. and Maki introduced and he studied generalized a-closed maps. [3] – [7] A study on soft α−open sets was studied by D. Sivaraj and V.E. Sasikala. [8] Basavaraj M. Ittanagi & Veeresha A. Sajjanar studied and developed weakly semi CS in topological spaces in 2017. [9, 10] Levine, N., looked on research on semi-open sets and semi-continuity in topological spaces. [11] Gnanambal, Y., discussed and investigated on general pre-regular closed sets in topological spaces. [12] Generalized separation axioms in bitopological spaces title was developed by Tantawy, O.A. El. and Abu Donia, H.M., in (2005). [13] S. Sam Sharmila Shofia, V. E. Sasikala, created and explored Delta generalized closed sets in topological spaces. [14] M. Munshi studied on separation axioms. [15] Indira, T. & Rekha, K., introduced and studied the applications of *b-open sets and* *b-open sets in topological spaces in 2012. [16] Ashish Kar and P. Bhattacharyya introduced and studied on some weak separation axioms. [17] S.N. Maheshwari and R. Prasad studied on some new separation axioms. [18] A separation axiom between semi-T_0 and semi-T_1 was developed by Caldas**,** M. [19] Dorsett**, C.,** studied research on semi-compactness, semi-separation axioms, and product spaces. [20] Jafari**,** S**.,** On a discussed and studied weak separation axiom. [21] D. Sivaraj and A.P. Ponraj, studied on, on soft semi weakly generalized closed set in soft topological spaces.[22] S. Saranya, V. E. Sasikala, was developed by " Compactness and Connectedness in Beta weakly semi-closed sets in Topological space. [23] S. Saranya, V. E. Sasikala, looked on research on The collection of various βws-closed sets in Topological space. The research aims to introduce the idea of extending separation axioms through closed sets in topological through within TS, along with providing characterizations for these concepts.

ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

2. PRELIMINARIES :

Throughout this paper (X,τ), (Y,σ) (or simply X and Y) represents topological spaces on which separation axioms are assumed unless otherwise mentioned. For a subset A of (X, τ) A of (X, τ) , cl (A) and int (A) and int (A) denote the closure of A and interior of A respectively .

Definition 2.1: Let (X, τ) be a topological space. A subset A of (X, τ) is called

(i) α-open set if A ⊆ int(cl(int(A))) and α-closed set if cl(int(cl(A))) ⊆ A.

(ii) Semi open set if A ⊆ cl(int(A)) and semi closed set if int(cl(A)) ⊆ A.

(iii) Pre-open set if A ⊆ int(cl(A)) and pre-closed set if cl(int(A)) ⊆ A.

(iv) Semi pre-open (β-open) set if A ⊆ cl(int(cl(A))) and semi pre closed set if int(cl(int(A))) ⊆ A.

(v) Regular open set if A = int (cl(A)) and a regular closed set if cl(int(A)) = A

The complement of the above mentioned closed sets are their respective open sets.

Let (X, τ) be a topological space. The intersection of all closed sets containing A is called closure of A and is denoted by cl(A). The union of all open sets contained in A is called interior of A and is denoted by int(A).

Definition 2.2: A subset A of a space X is called beta weakly semi-closed set (Briefly, βws-closed set), if $\beta c(A) \subseteq$ U whenever A ⊆ U and U is ws-open set.

Definition: 2.3: Let P be a subset of a TS in Ӽ, βws-interior of P be symbolized by βwsint (P) it was described as βwsint (P) = ∪{Q⊆P and Q be βws-OS in Ӽ} or ∪{Q: Q⊆P and Q is βwsO(Ӽ) or βwsint (P)} be a combination of all βws-OS with includes P.

Definition: 2.4: Assume P is a subset of a TS in Ӽ**,** Beta weakly semi –closure on P be denoted by βwscl (P) also it is defined as βws cl (P) = ∩{Q: P⊆Q, Q is βws-CS in Ӽ} or intersection on all βws –CS in Ӽ.

Definition: 2.5 : If X is a TS with the ӽ∈Ӽ. If subset N of Ӽ it is said βws – Neighborhood (briefly - βws – nbd) of ӽ if ∃an βws – OSG that the ӽ∈ɕ⊆N.

Definition 2.6 : A collection {A_i: i∈I} of βws-open sets in a topological space X is called βws - open cover of a subset A in X if A $\subseteq \bigcup_{i \in I} A_i$.

Definition 2.7 : A topological space X is called βws -compact if every βws-open cover of X has a finite subcover.

Definition 2.8 : A subset A of a topological space X is called βws-compact relative to X if for every collection {A_i: i∈I} of βws -open subsets of X such that A$\subseteq \bigcup_{i \in I} A_i$ there exists a finite subset I_0 of I such that A$\subseteq \bigcup_{i \in I_0} A_i$.

Definition 2.9 : A subset A of a topological space X is called βws-compact if A is βws-compact of the subspace of X.

Definition 2.10: A topological space X is said to be countably βws-compact if every countable βws-open cover of X has a finite subcover.

Definition:2.11: A topological space X is said to be α^*-T_0 if for each pair of distinct points of X, there exists a α^*- open set containing one of the points but not the other.

Definition 2.12: A topological space X is said to be $\alpha^* - T_1$ space if for each pair of distinct points x, y of X, there exists a pair of α^*- open sets, one containing x but not y and the other containing y but not x.

Definition 2.13 : A space X is said to be $\alpha^* - T_2$ Spaces **,** if for each pair of distinct points x, y

ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

of X, there exists disjoint α^* - open sets U and V such that $x\in U$ and $y\in V$.

3. βws - T_o spaces :

In this section, we introduce the concept of $\beta ws - T_0$ space (i = 0,1,2..) and obtained some of their properties.

Definition 3.1: A topological space X is said to be βws - T_o space if for each pair of distinct points x, y of X, there exists an βws - open set containing one point but not the other.

Theorem 3.2: Every space is a βws - T_o space.

Proof: Let X be a βws - T_o space. Let x, y be two distinct points in X. Since X is βws - T_o space, there exists a open set M in X such that $x\in X$, $y\notin X$. Since, every open set is βws - open, X is βws - open in x. Thus, for any two distinct points x, y in X there exists an βws - open X such that $x\in X$, $y\notin X$. Hence X is a βws - T_o space.

Example : 3.3 Let X = {a, b, c, d}, τ = {x, ϕ , {a},{b},{a, d},{b, c},{a, b},{a, b, c},{a, b, d}}. Then βws – closed set are {X, ϕ, {b},{c},{d},{b, c},{c, d},{b, d}}. And βws – open set are {X, ϕ, {a, c, d},{a, b, d},{a, b, c},{a, d},{a, b},{a, c}}. Let x = {b} and y = {a, c, d} where $x\in X$, $y\notin X$ and $x \neq y$. If for each pair of distinct points x, y of X, there exists an βws - open set containing one point but not the other. Hence, it is an βws - T_o space.

Remark 3.4: Clearly, every ws -T_o space is a βws -T_o space. Since, every open set is βws - open but the converse is not true.

Theorem 3.5: A space X is a βws - T_o space if and only if βws - closures of distinct points are distinct.

Proof: Let x, $y\in X$ with $x \neq y$ and X be a βws - T_o space. We shall show that βwscl({x}) ≠ βwscl({y}). Since, X is βws - T_o, there exists a βws - open M such that $x\in M$, $y\notin M$. Also, $x \notin X$-M and $y\in X$-M, where X-M is βws - closed set in X. Since, βws-cl ({y}) is the intersection of all βws-closed sets which contain y. Hence $y\in$βwscl({y}) but $x\notin$βwscl({y}) as $x\notin X$-M. Therefore, βwscl({x}) ≠ βwscl({y}).

Conversely, suppose that for any pair of distinct points $x,y\in X$, βwscl({x}) ≠ βwscl({y}). Then, there exists atleast one point $z\in X$ such that $z\in$βwscl({x}) but $z\notin$βwscl({y}). We claim that $x\notin$βws - cl({y}). If $x\in$βwscl({y}) then βwscl({x}) ⊆ βwscl({y}). So, $z\in$βwscl({x}) which is a contradiction. Now, $x\notin$βwscl({y}) implies $x\in X$ - βwscl({y}) which is a βws - open set in X containing x but not y. Hence, X is βws - T_o space.

Theorem 3.6: Let f: (X, τ)→ (Y, σ) be a bijection, βws - open map and X is βws - T_o space, then Y is also βws - T_o space.

Proof: Let Y_1, $Y_2\in Y$ with $Y_1\neq Y_2$ **.** Since f is a bijection, there exists X_1, $X_2\in$ X with $X_1\neq X_2$ such that $f(X_1) = Y_1$ and $f(X_2) = Y_2$. Since, X is βws- T_0 , there exists a βws – open set M in X such that an $x_1\in M$, $x_2\notin M$ open set in Y. Now, we have $X_1\in M \Rightarrow f(X_1)\in f(M) \Rightarrow Y_1\in f(M)$ and $Y_2\notin f(M)$. Hence, for any two distinct points $Y_1, Y_2\in Y$, there exists βws-open set f(M) in Y such that $Y_1\in f(M)$ and $Y_2 \notin f(M)$. Hence Y is a βws- T_0 space.

Theorem 3.7 : Let f: (X,τ) → (Y, σ) be a bijection, βws-irresolute and Y is βws-T_0 space, then X is also βws-T_0 space.

Proof: Let Y_1, $Y_2 \in Y$ with $Y_1 \neq Y_2$**.** Since f is a bijection, there exists X_1, $X_2 \in X$ with $X_1\neq X_2$ such that $f(X_1) = Y_1$ and $f(X_2) = Y_2 \Rightarrow X_1= f^{-1} (y_2)$ and $X_2=f^{-1}(y_2)$. Since, Y is βws -T_0, there

ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

exists a βws - open set M in Y such that $Y_1 \in M$, $Y_2 \notin M$. Since, f is βws- irresolute map, $f^{-1}(M)$ is a βws-open set in X. Now, we have $Y_1 \in M \Rightarrow f^{-1}(y_1) \in f^{-1}(M) \Rightarrow X_1 \in f^{-1}(M)$ and $y_2 \notin M \Rightarrow f^{-1}(Y_2) \notin f^{-1}(M) \Rightarrow X_2 \notin f^{-1}(M)$. Hence, for any two distinct points X_1, $X_2 \in$ X, there exists βws-open set $f^{-1}(M)$ in X such that $X_1 \in f^{-1}(M)$ and $X_2 \notin f^{-1}(M)$. Hence X is βws - T_0 space.

4. βws -T_1 space:

Definition 4.1: A topological space X is said to be βws -T_1 space if for each pair of distinct points x, y of X, there exists a pair of βws - open sets, one containing x but not y and the other containing y but not x.

Remark 4.2: Every βws - T_1 space is βws- T_0 space but the converse is not true.

EXAMPLE 4.3: Let X = {a, b, c, d }, τ = {x, ϕ, {a},{c},{d},{a, c, d},{c, d},{a, d},{a, c}}. Then βws- closed sets = {x, ϕ, {a},{b},{c},{a, b},{b, c},{a, c}}. And βws- open sets = {x, ϕ, {b, c, d},{a, c, d},{a, b, d},{c, d},{a, d},{b, d}}. Let U= x = {c, d}, V = y = {a, d} where x, y ∈X and x∈U, y∉U also y∈V and x∉V. It is βws - T_1 space.

Theorem 4.4: For a topological space X the following are equivalent.

(i) X is βws - T_1 space.

(ii) For every x∈X, {x} is βws - closed in X.

(iii) Each subset of X is the intersection of βws - open sets containing it.

(iv) The intersection of all βws - open sets in X containing the points x is {x}.

Proof: (i) ⇒(ii) Suppose that X is βws - T_1 space. Let x∈X. Then for every y≠x, there exists a βws-open sets U in X containing y but not x, U∩{x}≠ ϕ. Therefore, x∈U a contradiction. Thus, x is βws - closed.

(ii) ⇒(iii): Let A⊆X. Then, for each x∈X\A, {x} is βws - closed in X and hence, X \ {x} is βws - open. Clearly, A⊆X \ {x} for each x∈X\A.

Therefore, A = ∩ { X/ {x} : x∈X\A} which proves (iii).

(iii) ⇒(iv): Taking A = {x}, by (iii) A= {x} = ∩{ U: U is βws - open and x∈U}.

This proves (iv).

(iv) ⇒(i): Let x,y ∈X with y ≠ x. Then y∉{x} = ∩{U: U is βws - open and x∈U}. Hence, there exists a βws - open set U containing x but not y. Similarly, there exists a βws - open set V containing y but not x. Thus, X is βws- T_1 space.

5. βws – T_2 Space :

Definition 5.1: A topological space (x, τ) is said to be βws - T_2 space if for each pair of any points x, y ∈ X, such that x≠y there exists βws - open sets U; V⊆ x such that x∈U, y∈V and U∩V = ϕ respectively.

EXAMPLE: 5.2: Let X = { a, b, c, d}, τ = { x, ϕ, {a},{c},{b},{a, b},{b, c},{a, c},{a, b, c}} The complement of the open sets are their respective closed set are {X, ϕ, {b, c, d},{a, b, d},{a, c, d},{c, d},{a, d},{b, d},{d}}. Then βws closed set are {X, ϕ, {b, c, d},{a, b, d},{a, b, c},{b, d} {b, c},{a, b}} and βws open sets (V) = {X, ϕ, {a},{c},{d},{a, c} {a, d},{c, d}}. Let U = *x* = {a}, V = y = {d}, where *x*∈U, y∈V and U∩V = ϕ.

Theorem 5.3: If (x, τ) is βws - T_2 space then for x≠y∈X there exists a βws – open set U such that x∈U and y∉βws cl (U).

ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

Proof : Let x, y be distinct points of X. Since X is βws - T_2 there exists disjoint βws – open sets U and V in X such that x∈U and y∈V. Therefore V^c is βws – closed set such that βwscl(U) ⊆ V^c. Since y∈V, we have y∉V^c. Thus y∉βws cl (U).

Theorem 5.4: Let f: (X, τ) → (Y, σ) be a bijection.

(i) If f is a βws - continuous and Y is T_2, then X is βws - T_2 .

(ii) If f is a βws - irresolute and Y is βws -T_2, then X is βws - T_2 .

(iii) If f is a βws - open and X is T_2, then Y is βws - T_2.

Proof: (i) Suppose f is βws - continuous bijection and Y is T_2. Let x_1 , x_2 ∈ X, with $x_1 \neq x_2$. Let $Y_1 = f(x_1)$ and $Y_2 = f(X_2)$. Since, f is one to one, $Y_1 \neq Y_2$. Since, Y is T_2, there exists an open set U and V containing Y_1 and Y_2 respectively. Since, f is βws - continuous bijection, and $f^{-1}(U)$ $f^{-1}(V)$ are disjoint βws-open sets in X containing x_1 and x_2 respectively. Thus X is βws $- T_2$.

(ii) Proof is similar to (i).

(iii) Suppose f is βws - open bijection and x is T_1. Let $y_1 \neq y_2$ ∈Y. Since f is bijection, there exist x_1, x_2 in X such that y_1= f(x_1) and y_2= f(x_2) with $x_1 \neq x_2$. Since X is T_2, there exist disjoint open sets U and V in X such that x_1∈U and x_1∈V. Since, f is βws - open in Y such that $Y_1 = f(X_1) \in f(U)$ and Y_2= f(X_2) with x_1 and x_2 .Since, f is a bijection, y_2= f(x_2)∉f(U) and y_1= f(x_1)∉ f(V). Thus, y is βws $- T_2$.

6. CONCLUSION:

In this chapter, the concept of βws- T_0 (i = 0,1,2 …) sets in topological spaces are described and some new form of separation axioms using have been studied and discussed. The primary goal of this work was to get a deeper understanding of the consequences of Separation axioms through in Beta weakly semi-CS in TS and suggest future lines of research. We have obtained several noteworthy results from our analysis of Beta weakly semi closed sets. The study has strengthened the theoretical foundations of these concepts by developing and verifying several theorems that show these ideas may hold true in various topological domains. There are a number of interesting avenues for further research in the future. The application separation axioms through in Beta weakly semi-CS in TS. The study of Separation axioms through in Beta weakly semi-CS in TS has led to significant discoveries in the TS.

7. BIBLIOGRAPHY:

1. Alias B. Khalaf and Suzan N. Dawod, (2013), g*b - separation axioms, Gen. Math. Notes, 16, pp :14 - 31.
2. Devi, R. Balachandran, K. and Maki, H. (1998), Generalized a - closed maps and a - generalized closed maps, Indian J. Pure. Appl. Math., 29, pp : 37 - 49.
3. D. Sivaraj and V.E. Sasikala, ‘A Study on Soft α−open sets’, IOSR Journal of Mathematics, vol.12,pp.70-74.,issue.5.
4. V. Kavitha, V.E. Sasikala (Corresponding author), Beta Generalized CS in Topological Spaces, Journal of Algebraic Statistics, (Impact factor 0.21), ISSN: 1309-3452, Volume 13, No. 3, July, 2022 pp. 891-898, ISSN: 1309-3452.

ISSN:2731-538X | E-ISSN:2731-5398
Vol. 19 No. 01 (2025)

5. V.E. Sasikala and D. Sivaraj, A Study on Soft Pre-open Sets, International Journals of Mathematical Archive, (Impact Factor: 5.09), ISSN 2229-5046, Vol.8, Issue.3, , pp.138-143, 2017.
6. V.E. Sasikala, D. Sivaraj and A.P. Ponraj, Soft swg Separation Axioms in Soft Topological Spaces, Journal of Applied Science and Computations, (Impact Factor.5.8), ISSN No. 1076-5161, Vol.6, Issue No.5, Year.2019, pp. 2236-2245. (UGC)
7. D.Sivaraj and V.E. Sasikala, ‘A Study on Soft α−open sets’, IOSR Journal of Mathematics, vol.12,pp.70-74.Vol.12,issue.5,2016 .
8. Basavaraj M.Ittanagi & Veeresha A Sajjanar, 2017, ‘On weakly semi CS in topological Spaces’, int. J. Math. Arch., vol. 8, No. 9, pp. 126 - 134. ISSN 2229 – 5046
9. Levine, N. (1963), Semi open sets and semi continuity in topological spaces, Amer. Math. Monthly, 70, pp:36 - 41.
10. Levine, N. (1970), Generalized closed sets in topology, Rend. Circ. Math. Palermo, 19, pp:89 - 96.
11. Gnanambal, Y. (1997), On generalized pre regular closed sets in topological spaces, Indian J. Pure. Appl. Math., 28, pp:351 - 360.
12. Tantawy, O.A.El. and Abu Donia, H.M., (2005), Generalised separation axioms in bitopological spaces, The Arabian J. for Sci. and Engg., 30, 117-129.
13. S. Sam Sharmila Shofia, V. E. Sasikala, Delta Generalized Closed Set in Topological Spaces, Design Engineering(Toronto), ISSN: 0011-9342 , Year 2021, Issue: 9, Volume: 2021, Pages: 10853 – 10861
14. M. Munshi, Separation axioms, Acta ciencia indica, issue no 12 ,pp:140- 144,1986.
15. Indira, T & Rekha, K 2012, ‘Applications of *b-open sets and **b-open sets in Topological spaces’, Annals of Pure and Applied Mathematics, vol.1, no.1,pp. 44-56.
16. Ashish Kar and P. Bhattacharyya , Some weak separation axioms, Bull. Calcutta Math. Soc., 82, pp : 415-422.1990.
17. S. N. Maheshwari and R. Prasad, Some new separation axioms, Annales de la Soc.Sci.de Bruxells,T. pp:395-402,1975.
18. **Caldas, M.,**'A separation axiom between semi-T_0 and semi-T_1',Mem. Fac. Sci. Kochi Univ. (Math.), Vol.18, pp. 37-42,1997.
19. **Dorsett, C.,**'Semi compactness, semi-separation axioms, and product spaces', Bull. Malays. Math. Sci. Soc., vol. 2, No. 4, pp. 21-28,1981.
20. **Jafari, S.,**'On a weak separation axiom', Far East J. Math. Sci., vol. 3, no. 5, 2001, Pp: 779-789.
21. V.E. Sasikala, D. Sivaraj and A.P. Ponraj, “On soft semi weakly generalized closed set in soft topological spaces”, International Journal of Innovative Technology and Exploring Engineering, Vol.8, No.12, pp.1252-1256, 2019.
22. S. Saranya, V. E. Sasikala, “ Compactness and Connectedness in Beta weakly semi-closed sets in Topological space”, Communication on Applied Nonlinear Analysis, ISSN :1074-133X, vol .31, no. 5s ,2024.
23. S. Saranya, V. E. Sasikala, The collection of various βws-closed sets in Topological space”, Advances in Nonlinear varaiational Inequalities, ISSN: 1092-910x, vol 28, no .1s (2025).